The spiral of Theodorus and sums of zeta-values at the half-integers

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ABSTRACT. The total angular distance traversed by the *spiral of Theodorus* is governed by the *Schneckenkonstante* K introduced by Hlawka. The only published estimate of K is the bound $K \leq 0.75$. We express K as a sum of Riemann zeta-values at the half-integers and compute it to 100 decimal places. We find similar formulas involving the Hurwitz zeta-function for the *analytic Theodorus spiral* and the *Theodorus constant* introduced by Davis.

1 Introduction

Theodorus of Cyrene (ca. 460–399 B.C.) taught Plato mathematics and was himself a pupil of Protagoras. Plato's dialogue *Theaetetus* tells that Theodorus was distinguished in the subjects of the quadrivium and also contains the following intriguing passage on irrational square-roots, quoted here from [12]:

> [Theodorus] was proving to us a certain thing about square roots, I mean of three square feet and of five square feet, namely that these roots are not commensurable in length with the foot-length, and he went on in this way, taking all the separate cases up to the root of 17 square feet, at which point, for some reason, he stopped.

It was discussed already in antiquity why Theodorus stopped at seventeen and what his method of proof was. There are at least four fundamentally different theories—not including the suggestion of Hardy and Wright that Theodorus simply became tired!—cf. [11, 12, 16].

One of these theories is due to the German amateur mathematician J. Anderhub, cf. [4, 14]. It involves the so-called *square-root spiral of Theodorus* or *Quadratwurzelschnecke*. This spiral consists of a sequence of points P_1, P_2, $P_3, \ldots$ in the plane circulating anti-clockwise around a centre P_0 such that $|P_0P_n| = \sqrt{n}$ and $|P_nP_{n+1}| = 1$ for all $n \geq 1$ (see Figure 1). Let θ_n be the angle $\angle P_nP_0P_{n+1}$. Then

$$\theta_n = \arctan \frac{1}{\sqrt{n}}$$

since $\angle P_0P_nP_{n+1}$ is a right angle. Further, let Θ_n be the total angular distance traversed by the spiral in $n-1$ steps, i.e.,

$$\Theta_n := \sum_{x=1}^{n-1} \theta_x. \tag{1}$$

Then $\angle P_1P_0P_n$ equals Θ_n modulo 2π. The spiral of Theodorus can thus alternatively be defined in the complex plane by $P_0 = 0$ and $P_n = \sqrt{n} \cdot \exp(\Theta_n i)$. It was Anderhub's discovery that $n = 17$ is the last value of n such that the spiral does not overlap, i.e., such that $\Theta_n < 2\pi$.

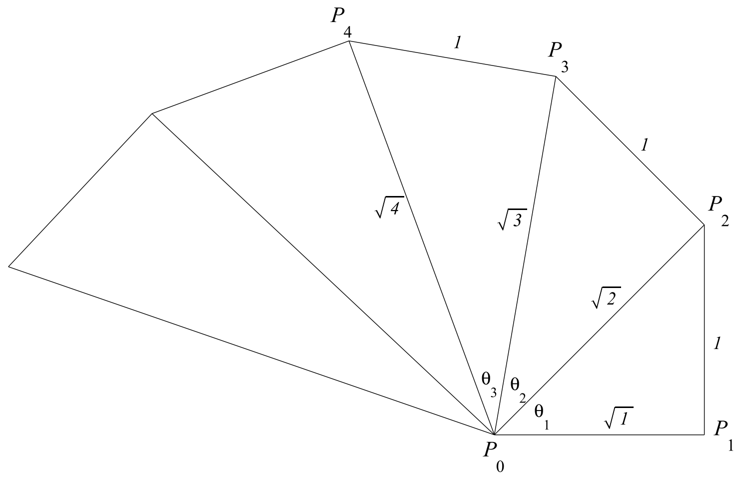

Figure 1.

2 An asymptotic formula and the Schneckenkonstante

Hlawka [14, eq. (13)] gives a formula for Θ_n of the form $\Theta_n = 2\sqrt{n}+K+$(terms of lower order) with a constant K which he terms *Schneckenkonstante*. However, the last coefficient in Hlawka's formula, which is also quoted in [4], seems to be incorrect. Hlawka moreover gives the bound $K \leq 0.75$ which, to the author's best knowledge, is the only published estimate of K. Better estimates appear in several unpublished manuscripts, see [5, 10] and the references in [7].

Theorem 1. *The angular distance traversed by the spiral of Theodorus satisfies the asymptotic formula*

$$\Theta_n = 2\sqrt{n} + K + \frac{1}{6\sqrt{n}} - \frac{1}{120n\sqrt{n}} - \frac{1}{840n^2\sqrt{n}} + \frac{5}{8064n^3\sqrt{n}} + \frac{1}{4224n^4\sqrt{n}} + O\left(\frac{1}{n^5\sqrt{n}}\right), \tag{2}$$

where K, Hlawka's Schneckenkonstante, is given by

$$K = \sum_{k=0}^{\infty} (-1)^k \frac{\zeta(k+\frac{1}{2})}{2k+1} = \frac{\pi}{4} + \sum_{k=0}^{\infty} (-1)^k \frac{\zeta(k+\frac{1}{2})-1}{2k+1} \tag{3}$$

or numerically

$$K = -2.1577829966\ 5944622092\ 9142786829\ 5777235041\ 3959860756$$
$$2455154895\ 5508588696\ 4679660648\ 1496694298\ 9463960898\ldots$$

Proof. The series

$$\arctan\frac{1}{\sqrt{x}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)x^{k+1/2}} \tag{4}$$

converges for $x \geq 1$. Hence, for any $N \geq 1$, one has

$$\begin{aligned}\Theta_n = \sum_{x=1}^{n-1}\arctan\frac{1}{\sqrt{x}} &= \sum_{x=1}^{n-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)x^{k+1/2}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\sum_{x=1}^{n-1}\frac{1}{x^{k+1/2}}\\ &= \sum_{k=0}^{N-1}\frac{(-1)^k}{2k+1}\sum_{x=1}^{n-1}\frac{1}{x^{k+1/2}} + \sum_{k=N}^{\infty}(-1)^k\frac{\zeta(k+\frac{1}{2})}{2k+1} + O\left(\frac{1}{n^{N-1/2}}\right) \end{aligned}\tag{5}$$

for $n \to \infty$, where ζ is Riemann's zeta-function. For any complex exponent s and any positive integers m and n, Euler's summation formula [8, p. 469] gives

$$\begin{aligned}\sum_{x=1}^{n-1}x^s = \int_1^n x^s dx - \tfrac{1}{2}(n^s - 1) + \sum_{t=2}^{m}\frac{B_t}{t!}s^{(t-1)}(n^{s+1-t} - 1)\\ + (-1)^{m+1}\int_1^n\frac{B_m(\{x\})}{m!}s^{(m)}x^{s-m}dx.\end{aligned}\tag{6}$$

Here, B_t and $B_m(x)$ are Bernoulli numbers and polynomials, $\{x\}$ is the fractional part of x, and $s^{(t)}$ is the falling factorial $s(s-1)\cdots(s-t+1)$. It is only necessary to sum over the even values of t since $B_t = 0$ for odd $t > 1$. For $s \neq -1$ and $m > \Re(s)+1$, (6) can be rewritten as

$$\sum_{x=1}^{n-1}x^s = C(s) + \frac{1}{s+1}n^{s+1} - \tfrac{1}{2}n^s + \sum_{t=2}^{m}\frac{B_t}{t!}s^{(t-1)}n^{s+1-t} + O(n^{s-m}) \tag{7}$$

for $n \to \infty$, where all terms independent of n have been collected in the constant

$$C(s) = -\frac{1}{s+1} + \tfrac{1}{2} - \sum_{t=2}^{m}\frac{B_t}{t!}s^{(t-1)} + (-1)^{m+1}\int_1^{\infty}\frac{B_m(\{x\})}{m!}s^{(m)}x^{s-m}dx. \tag{8}$$

It follows from (7) that $C(s)$ is independent of m, and also that

$$C(s) = \sum_{x=1}^{\infty}x^s = \zeta(-s) \quad \text{for } \Re(s) < -1.$$

It follows from (8) by Leibniz's integral rule that $C(s)$ is an analytic function of s. Consequently, $C(s)$ and $\zeta(-s)$ agree for all complex $s \neq -1$ by analytic continuation. Thus, for example,

$$\sum_{x=1}^{n-1}\frac{1}{\sqrt{x}} = 2\sqrt{n}+\zeta(\tfrac{1}{2}) - \frac{1}{2\sqrt{n}} - \frac{1}{24n\sqrt{n}} + \frac{1}{384n^2\sqrt{n}} - \frac{1}{1024n^3\sqrt{n}} + \frac{143}{163840n^4\sqrt{n}} + O\left(\frac{1}{n^5\sqrt{n}}\right), \tag{9}$$

and similarly for $s = -\frac{3}{2}, -\frac{5}{2}$, etc. Inserting these formulas into (5) with $N = 4$ gives (2) and the first equality of (3); the second equality follows from Leibniz's formula

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots$$

For computational purposes, the second series in (3) is much superior to the first since $\zeta(x) - 1 \sim 2^{-x}$ for real $x \to \infty$. Thus, the first 322 terms give the 100 decimal places stated. $\square$

Figure 2 shows the spiral of Theodorus together with the curve with polar coordinates $r(t) = t$ and $\varphi(t) = 2t + K + \frac{1}{6}t^{-1}$, $t > 0$, and the points on that curve corresponding to $t = \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots$ As it appears, the first three terms of (2) approximate Θ_n very well.

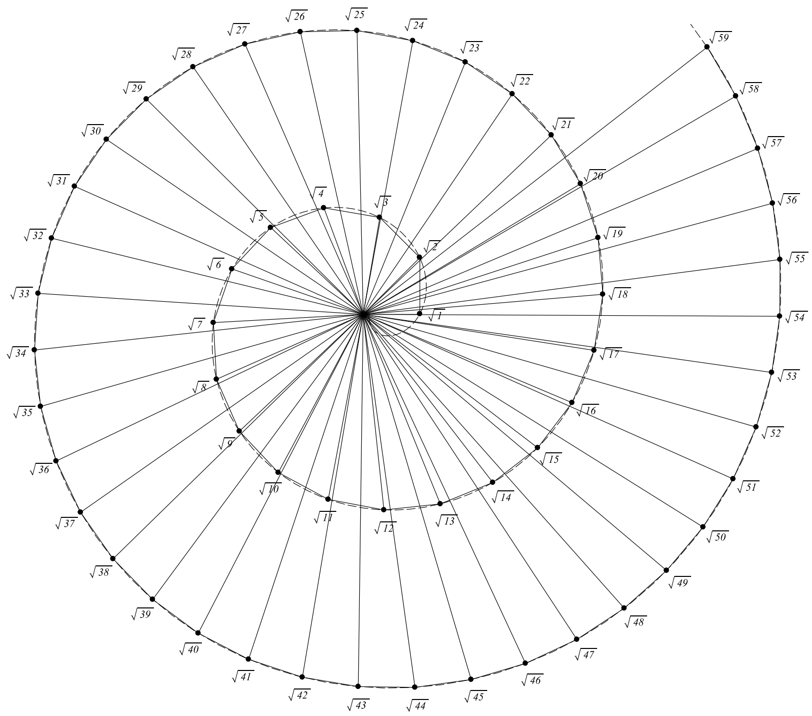

Figure 2.

Comparing (2) with the equivalent expression

$$\Theta_{n+1} = 2\sqrt{n} + K + \frac{7}{6\sqrt{n}} - \frac{41}{120n\sqrt{n}} + \frac{167}{840n^2\sqrt{n}} - \frac{1147}{8064n^3\sqrt{n}} + \frac{1411}{12672n^4\sqrt{n}} + O\left(\frac{1}{n^5\sqrt{n}}\right) \quad (10)$$

suggests that, for reasons unknown to the author, (1) is the "right" definition of Θ_n rather than $\sum_{x=1}^{n} \theta_x$.

It is worth noting that letting s be a nonnegative integer in (6) and adding n^s to each side gives

$$\sum_{x=1}^{n} x^s = \frac{1}{s+1} n^{s+1} + \frac{1}{2} n^s + \sum_{t=2}^{s+1} \frac{B_t}{t!} s^{(t-1)} n^{s+1-t} + \zeta(-s).$$

These are the power sum polynomials known already to Jakob Bernoulli (who was, incidentally, the inventor of another spiral, the *spira mirabilis*). A slightly modified application of Euler's summation formula shows that these polynomials have no constant terms. Hence, comparison with the constant term of the above formula gives the zeta-values at the nonpositive integers:

$$\zeta(0) = -\tfrac{1}{2} \quad \text{and} \quad \zeta(-s) = -\frac{B_{s+1}}{s+1} \quad \text{for} \quad s > 0.$$

We mention some alternative approaches to the numerical computation of K. Using Euler's summation formula on θ_x directly gives

$$K = -1 - \frac{3\pi}{8} - \int_1^\infty \frac{\{x\} - \frac{1}{2}}{2\sqrt{x}(x+1)} dx, \tag{11}$$

but this integral representation hardly gives K to more than a few places. Another way to calculate K is to isolate it in (2) as

$$K \approx \Theta_n - 2\sqrt{n} - \frac{1}{6\sqrt{n}} + \cdots \tag{12}$$

Taking, say, $n = 10^4$ and including 400 terms on the right gives K to more than 1000 places. Finally, subtracting (9) from (2) gives

$$K = \zeta\left(\tfrac{1}{2}\right) + \sum_{x=1}^{\infty} \left(\arctan \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}} \right). \tag{13}$$

Although this sum converges rather slowly compared to the second sum of (3), it can be computed very efficiently using convergence acceleration techniques such as Levin's u-transform [15].

3 Uniform distribution

Figure 2 above suggests that the sequence (Θ_n) is uniformly distributed modulo 2π. Using $\Theta_n = 2\sqrt{n} + K + o(1)$ and a theorem of Fejér, Hlawka [13, 14] gives a short proof of this fact (originally due to W. Neiss). We note that there is an even shorter proof using a generalization of Fejér's theorem due to van der Corput [3]. This result states that a real sequence (x_n) is uniformly distributed to any modulus if the sequence of differences $\Delta x_n := x_{n+1} - x_n$ satisfies $\Delta x_n \to 0$ monotonically and $n \cdot \Delta x_n \to \pm\infty$. In our case, $\Delta\Theta_n = \arctan n^{-1/2} \sim n^{-1/2}$ clearly satisfies the conditions of van der Corput's theorem.

4 The analytic Theodorus spiral and the Theodorus constant

Davis [4] defines an *analytic Theodorus spiral* with polar coordinates

$$r(t) = \sqrt{t}\ , \quad \varphi(t) = \sum_{x=1}^{\infty} \left(\arctan \frac{1}{\sqrt{x}} - \arctan \frac{1}{\sqrt{x+t-1}} \right) \tag{14}$$

for real $t > 0$. If n is a positive integer, then clearly $\varphi(n) = \sum_{x=1}^{n-1} \arctan x^{-1/2} = \Theta_n$. Gronau [9] shows a uniqueness result for $\varphi(t)$ similar to the Bohr-Mollerup theorem on the Gamma-function. The angular velocity is obtained by term-wise differentiation of (14) as $\varphi'(t) = \frac{1}{2}U(t)$ with

$$U(t) = \sum_{x=1}^{\infty} \frac{1}{(x+t)\sqrt{x+t-1}}.$$

Davis defines the *Theodorus constant* as the attractive sum

$$T = U(1) = \sum_{x=1}^{\infty} \frac{1}{(x+1)\sqrt{x}} = 1.8600250792\ldots$$

and computes it to 10 places. In an interesting analysis, Gautschi [4, 7] shows how T can be computed to higher precision using Gaussian quadrature.

Theorem 2. *The angular coordinate of Davis's analytic Theodorus spiral satisfies*

$$\varphi(t) = K - \arctan\frac{1}{\sqrt{t}} - \sum_{k=0}^{\infty}(-1)^k \frac{\zeta(k+\frac{1}{2}, t+1)}{2k+1} \tag{15}$$

for real $t > 0$, where $\zeta(s,q)$ is the Hurwitz zeta-function, and K is the Schneckenkonstante. The Theodorus constant is given by the quickly converging series

$$T = \tfrac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k+1}\left\{\zeta(k+\tfrac{1}{2}) - 1\right\}. \tag{16}$$

Proof. For real $q > 0$ and complex $s \neq 1$, the Hurwitz zeta-function $\zeta(s,q)$ is defined by

$$\zeta(s,q) = \sum_{x=0}^{\infty} \frac{1}{(q+x)^s}$$

for $\Re(s) > 1$ and for all $s \neq 1$ by analytic continuation. Thus, $\zeta(s,1)$ equals the usual Riemann zeta-function $\zeta(s)$. Similarly, $\zeta(s,2)$ equals $\zeta(s) - 1$, and

$$\sum_{x=2}^{\infty}\left(\frac{1}{x^s} - \frac{1}{(x+t-1)^s}\right) = \zeta(s,2) - \zeta(s,t+1) \tag{17}$$

for all $s \neq 1$, again by analytic continuation. Recall that (4) converges for $x \geq 1$. Now,

$$\begin{aligned}
\varphi(t) &= \sum_{x=1}^{\infty}\left(\arctan\frac{1}{\sqrt{x}} - \arctan\frac{1}{\sqrt{x+t-1}}\right) \\
&= \frac{\pi}{4} - \arctan\frac{1}{\sqrt{t}} + \sum_{x=2}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(\frac{1}{x^{k+1/2}} - \frac{1}{(x+t-1)^{k+1/2}}\right) \\
&= \frac{\pi}{4} - \arctan\frac{1}{\sqrt{t}} + \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\sum_{x=2}^{\infty}\left(\frac{1}{x^{k+1/2}} - \frac{1}{(x+t-1)^{k+1/2}}\right) \\
&= \frac{\pi}{4} - \arctan\frac{1}{\sqrt{t}} + \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left\{\zeta(k+\tfrac{1}{2},2) - \zeta(k+\tfrac{1}{2},t+1)\right\} \\
&= K - \arctan\frac{1}{\sqrt{t}} - \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\zeta(k+\tfrac{1}{2},t+1),
\end{aligned}$$

where we have used (3), (17), the identity $\zeta(s,2)=\zeta(s)-1$, and the (easily seen) fact that the double sum in the second line converges absolutely.

The series

$$\frac{1}{(x+1)\sqrt{x}}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{x^{k+1/2}}$$

converges for $x>1$. As before we get

$$\begin{aligned}
U(t) &= \sum_{x=1}^{\infty}\frac{1}{(x+t)\sqrt{x+t-1}}\\
&= \frac{1}{(t+1)\sqrt{t}}+\sum_{x=2}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(x+t-1)^{k+1/2}}\\
&= \frac{1}{(t+1)\sqrt{t}}+\sum_{k=1}^{\infty}(-1)^{k+1}\sum_{x=2}^{\infty}\frac{1}{(x+t-1)^{k+1/2}}\\
&= \frac{1}{(t+1)\sqrt{t}}+\sum_{k=1}^{\infty}(-1)^{k+1}\zeta(k+\tfrac{1}{2},t+1)
\end{aligned}$$

for $t>0$. Letting $t=1$ gives $T=U(1)$. □

5 Sums of zeta-values at the integers and half-integers

There is a vast literature on sums of zeta-values at the integers, cf. [1, 2]. To give some elementary examples, one has

$$\sum_{k=2}^{\infty}\{\zeta(k)-1\}=1,\qquad \sum_{k=2}^{\infty}(-1)^k\{\zeta(k)-1\}=\tfrac{1}{2},$$

$$\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k}=1-\gamma,\qquad \sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)-1}{k}=\gamma-1+\log 2,$$

where γ is Euler's constant. Other classical constants such as π, Mertens's constant, Hardy-Littlewood's twin prime constant, Khinchin's constant, and Landau-Ramanujan's constant are expressed as sums or products of zeta-values at the integers in [6].

In contrast, the author has been unable to find a closed-form expression for *any* sum of zeta-values at the half-integers such as (3) and (16). To conclude, we mention without proof

$$\sum_{x=2}^{\infty}\frac{1}{(x-1)\sqrt{x}}=\sum_{k=1}^{\infty}\left\{\zeta\left(k+\tfrac{1}{2}\right)-1\right\}=2.1840094702\ldots \tag{18}$$

and

$$\sum_{x=2}^{n}\operatorname{artanh}\frac{1}{\sqrt{x}}=2\sqrt{n}+K'+O\left(\frac{1}{\sqrt{n}}\right) \tag{19}$$

with

$$K'=\sum_{k=0}^{\infty}\frac{\zeta\left(k+\frac{1}{2}\right)-1}{2k+1}=-1.8265078108\ldots \tag{20}$$

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